# Homomorphism Theorems.

**Natural Homomorphism:** (Qazi- 82)

Let $H$ be a normal subgp of a group $G$, then the mapping $\varphi: G \longrightarrow G/H$ defined by $\varphi(g) = gH$ is called natural or canonical homomorphism of $G$ onto $G/H$.

**Remark:-** To every factor group $G/H$, there is a homomorphism $\varphi: G \longrightarrow G/H$ such that $\varphi(G) = G/H$.

**Theorem:** (Fundamental theorem of homomorphism) homomorphism theorem or 1st Isomorphism theorem)

Let $\varphi: G \longrightarrow G'$ be an epimorphism (onto homomorphism) from $G$ to $G'$. Then.

(a) the $K = \ker\varphi$ is normal subgroup of $G$.

(b) the factor group $G/K$ is isomorphic to $G'$ i.e $\varphi(G) \cong G/K$

(C) A subgroup $H'$ of $G'$ is normal in $G'$ iff its inverse image $H = \varphi^{-1}(H')$ is normal in $G$.

(d) There is one-one correspondence between the subgroups of $G'$ (Containing) and those sub-groups of $G$ which contain the kernel $K = \ker\varphi$.

**Proof:-**

(a) To show that $K = \ker\varphi = \{k \in G : \varphi(k) = e'\}$ is a normal subgroup of $G$, we first we show that $K$ is a subgroup of $G$.

Let $k_1, k_2 \in K$. Then

$$\varphi(k_1) = e' \quad \varphi(k_2) = e'$$

$$\text{and } \varphi(k_1 k_2^{-1}) = \varphi(k_1)(\varphi(k_2))^{-1}$$
$$= e' \cdot e'^{-1} = e' \cdot e' = e' \in G'$$

$\Rightarrow k_1 k_2^{-1} \in K$ and $K$ is sub-group.

Next we show that $K$ is normal in $G$.

For this let $k \in K$, $g \in G$. Then

$$\varphi(gkg^{-1}) = \varphi(g) \cdot \varphi(k) \cdot \varphi(g^{-1})$$
$$= \varphi(g) \cdot e' \cdot (\varphi(g))^{-1}$$
$$= e' \in G'$$

$\Rightarrow gkg^{-1} \in K$. Therefore $K$ is normal in $G$.

(b) We show that $G/K \cong G'$

For this define a mapping $\psi: G/K \longrightarrow G'$ as follows.

For any $gK \in G/K$, $g \in G$ we put

$$\psi(gK) = \phi(g)$$

Then for $g_1K, g_2K \in G/K$.

$$\psi((g_1K)(g_2K)) = \psi(g_1g_2K)$$
$$= \phi(g_1g_2)$$
$$= \phi(g_1)\psi(g_2) \quad (\because \phi \text{ is hom.})$$
$$= \psi(g_1K)\cdot\psi(g_2K)$$

Hence $\psi$ is a homomorphism and $G/K \cong G'$

(c) Suppose $H'$ is normal in in $G'$ and

$$H = \phi^{-1}(H') = \{h \in G : \phi(h) = h' \in H'\}$$

Since $K = \ker\phi$ is inverse image of $e'$

$\therefore$ $K$ is contained in $H$.

Let $h \in H$ and $g \in G$. Then $ghg^{-1} \in H$ ~~$\because H$ is normal in $G$~~

iff $\phi(ghg^{-1}) \in H'$ ~~$\therefore ghg^{-1} \in H$~~

But $\phi(ghg^{-1}) = \phi(g).\phi(h).\phi(g^{-1})$

$$= \phi(g).\phi(h).\phi(g)^{-1} \in H' \quad (\because H' \trianglelefteq G')$$

$\Rightarrow ghg^{-1} \in H$ and $H \trianglelefteq G$

Conversely suppose that $H$ is normal in $G$ and $H' = \phi(H)$

Let $h' \in H'$, $g' \in G'$

and $\phi(h) = h'$ $\phi(g) = g'$

$$g'h'g'^{-1} = \phi(g).\phi(h).\phi(g)^{-1}$$
$$= \phi(ghg^{-1})$$

$\because H \trianglelefteq G$

$ghg^{-1} \in H$

$\therefore$

$\Rightarrow \phi(ghg^{-1}) \in \phi(H) = H'$

$\Rightarrow g'h'g'^{-1} \in H'$ : Hence $H'$ is normal in $G'$

(d):- Let $\Omega$ be the collection of all sub-groups of $G$ containing $K$ and $\Omega'$ be the collection of all sub-groups of $G'$. Define a mapping $\alpha : \Omega \longrightarrow \Omega'$ by

$$\alpha(H) = \phi(H) = H'$$

If $H_1, H_2 \in \Omega$ and

$$\alpha(H_1) = \alpha(H_2) = H' \text{ (say)}$$

we show that $H_1 = H_2$.

Let $H_1 = \phi^{-1}(H')$

Then $H_1 \subseteq H$.

Let $h \in H$, $\phi(h) = h' = \phi(h_1)$

From $\alpha(H_1) = H' = \phi(H_1)$, for $h' \in H'$, $h_1 \in H_1$

Hence

$$h_1^{-1} h \in K$$

i.e. $h \in h_1 K \subseteq H_1$.

Thus $H \subseteq H_1 \Rightarrow H = H_1$

Similarly $H = H_2$.

Hence $\alpha$ is injective.

Also each $H' \in \Omega'$ is the image of an $H = \phi^{-1}(H')$. Hence $\alpha$ is surjective.

Thus $\alpha$ is one-one correspondence.

**Kernel of $\phi$:** Let $\phi : G \longrightarrow G'$ be a homomorphism. The set of all those elements of $G$ which are mapped onto the identity $e'$ of $G'$ is called the kernel of $\phi$ & is denoted by $\ker\phi$. Thus

$$\ker\phi = \{k \in G : \phi(k) = e'\} = \{k = \phi^{-1}(e')\}$$

**Theorem:-** Let $\phi : G \longrightarrow G'$ be a homomorphism of $G$ onto $G'$. Then

(a) If $H$ is a sub-group of $G$, then $\phi(H)$ is a sub-group of $G'$

(b) If $H$ is normal in $G$, $\phi(H)$ is normal in $G'$

**Proof:** (a) Since $\phi(e) \in \phi(H)$

$$\therefore \phi(H) \neq \phi.$$

Let $x_1, x_2 \in \phi(H)$

$\Rightarrow x_1 = \phi(h_1)$ $x_2 = \phi(h_2)$ for

Then $x_1x_2^{-1} = \varphi(h_1)(\varphi(h_2))^{-1} = \varphi(h_1).\varphi(h_2^{-1}) = \varphi(h_1h_2^{-1}) \in \varphi(H)$

$\Rightarrow \varphi(H)$ is a sub-group.

(b) Let $H \trianglelefteq G$

then $ghg^{-1} \in H \quad \forall g \in G, \forall h \in H$

Now any element of $G'$ is of form $\varphi(g)$ for some $g \in G$; and any element $\varphi(H)$ is of the form $\varphi(h)$ for som $h \in H$

Now

$$\varphi(g)(\varphi(h))\varphi(g)^{-1}$$
$$= \varphi(ghg^{-1}) \in \varphi(H) \quad \because ghg^{-1} \in H$$

Thus $\varphi(H) \trianglelefteq G'$.

**Theorem:** (Correspondence theorem)

Let $\varphi: G \longrightarrow G'$ be a homomorphism of $G$ onto $G'$, Then

(a) the preimage $H$ of a any sub-group $S$ of $G'$ is a sub-group of $G$ containing $\ker\varphi$

(b) If $S \trianglelefteq G'$, then $H \trianglelefteq G$. Furthermore if $H_1$ is any other sub-group of $G$ containing $\ker\varphi$ such that $\varphi(H_1) = S$, then $H_1 = H$.

**Proof:**

(a) $H = \{g \mid \varphi(g) \in S\}$

Since $\varphi(e)$ is identity of $G'$ and $S$ contains identity of $G'$, therefore $e \in H$. So $H \neq \phi$.

Let $h_1, h_2 \in H$

then $\varphi(h_1h_2^{-1}) = \varphi(h_1).(\varphi(h_2))^{-1} \in S \quad \because \varphi(h_1) \in S$
$\varphi(h_2)^{-1} \in S$

$\Rightarrow h_1h_2^{-1} \in H$

$\Rightarrow H$ is a sub-group of $G$.

Since $\ker\varphi = \{x \mid \varphi(x) = e' \in G'\}$ and $e' \in S$

$\therefore \ker\varphi \subseteq H$.

(b) Let $S \trianglelefteq G'$. We are to show that $H \trianglelefteq G$.

Let $h \in H, g \in G$

$\varphi(ghg^{-1}) = \varphi(g)\,\varphi(h)\,(\varphi(g))^{-1}$

Since $\varphi(g) \in G'$ and $\varphi(h) \in S$, $S \trianglelefteq G'$

Thus all the cosets are distinct

Let $g \in G$

Then $\phi(g) \in G'$

and so $\phi(g) \in Sg_i'$ for some integer $i$

$\Rightarrow \phi(g) = sg_i'$ , $s \in S$

Consider $x = gg_i^{-1}$

$\phi(x) = \phi(gg_i^{-1}) = \phi(g)\phi(g_i^{-1})$

$= sg_i'g_i'^{-1} = s$

$\Rightarrow gg_i^{-1}$ is the preimage of $S$

so that $gg_i^{-1} \in H$

$\Rightarrow g \in Hg_i$

$\Rightarrow$ Every element of $G$ is member of one of cosets $Hg_i : i = 1, 2, \dots n$. Hence these cosets constitute the totality of cosets of $H$ in $G$. Hence index of $H$ in $G$ is also $n$

**Corollary:-2:** Let $N \trianglelefteq G$ and $L$ be a subgroup of $G/N$. Then we can write $L = H/N$, where $H$ is a subgroup of $G$ containing $N$. If $L \trianglelefteq G/N$, then $H \trianglelefteq G$. If $H_1/N = H/N$ where $H_1$ & $H$ are sub-groups of $G$ containing $N$, then $H_1 = H$.

**Proof:-**

Let $f : G \longrightarrow G/N$ be the natural homomorphism and $H = \{g \mid f(g) \in L\}$

If $L \trianglelefteq G/N$

Then $H \trianglelefteq G$. (by correspondence theorem)

Since $f : g \longrightarrow Ng$

$\therefore f(H)$ consists of all cosets $Nh$, $h \in H$

$\Rightarrow f(H) = L$

Because $H \supseteq N = \ker f$ (by correspondence theorem)

Therefore $H/N$ makes sense and consists of all the cosets $Nh$, $h \in H$. Hence $H/N = f(H) = L$

Now if $H_1 \supseteq N$ and $H_1/N = H/N$

Then $f(H_1) = L = f(H)$

$\Rightarrow H_1 = H$ (by correspondence theorem)

$\therefore \quad \phi(g)\,\phi(h)\,(\phi(g))^{-1} \in S$

$\Rightarrow \phi(ghg^{-1}) \in S$

$\Rightarrow ghg^{-1} \in H \Rightarrow H \trianglelefteq G.$

Further let $H_1$ be sub-group of $G$ containing $\ker\phi$ and $\phi(H_1) = S$. We will show that

$$H_1 = H$$

Let $h_1 \in H_1$

$\Rightarrow \phi(h_1) \in S$

$\Rightarrow h_1 \in H$

$\Rightarrow H_1 \subseteq H \longrightarrow (A)$

If $h \in H$, then $\phi(h) = s \in S$

Choose $h_1 \in H_1$ such that

$$\phi(h_1) = s$$

Then $hh_1^{-1} \in \ker\phi \subseteq H_1$ and so $h \in H_1$

and $H \subseteq H_1 \longrightarrow (B)$

By (A) & (B)

$$H_1 = H$$

**Corollary:-1.** Let $\phi: G \longrightarrow G'$ be an onto homomorphism and $S$ be a subgroup of $G'$ of index $n < \infty$. Let $H$ be the preimage of $S$. Then $H$ is of index $n$ in $G$.

**Proof:-** Let $Sg_1', Sg_2', Sg_3', \ldots, Sg_n'$, where $g_i' \in G'$ be the distinct cosets of $S$ in $G'$. As $\phi$ is onto, there are $g_1, g_2, g_3, \ldots, g_n \in G$ such that

$$\phi(g_i) = g_i' \quad i = 1, 2, \ldots n$$

We claim that

$Hg_1, Hg_2, \ldots, Hg_n$ are distinct cosets of $H$ in $G$

Let $Hg_i = Hg_j$

Then $g_i g_j^{-1} \in H$

$\Rightarrow \phi(g_i g_j^{-1}) = \phi(g_i)(\phi(g_j))^{-1} \in S$

i.e. $g_i' g_j'^{-1} \in S$

$\Rightarrow Sg_i' = Sg_j'$

$\Rightarrow i = j$

So $Hg_i = Hg_i \Rightarrow g_i = g_j$

**Problem:** Let $\varphi: G \to G'$ be an onto homomorphism. Let $G'$ be a cyclic group of order 10. Prove that $G$ has normal sub-groups of index 2, 5, and 10

**Solution:-** Let $G' = gp(g')$. Then the sub-groups $G_1' = \{e\}$, $G_2' = gp(g^5)$, $G_3' = gp(g^3)$ are normal subgps of $G'$ of index 10, 5 and 2 respectively. Consequently their premages, by correspendence theorem are normal of index 10, 5 and 2 respectively.

**Problem:-** Let $H$ be a sub-group of index $n$ in $G$. Let $\varphi: G \to G'$ be an onto homomorphism. Prove that $\varphi(H)$ is of index $n$ in $G'$ if $H \supseteq \ker\varphi$.

**Solution:-** We only prove that $\varphi(H) = S$ is of finite index in $G'$, for then the result follows from corollary (4) of correspondence theorem.

If $Hg_1, Hg_2, \ldots\ldots Hg_n$ are the cosets of $H$ in $G$.
Then we show that $S(\varphi(g_1)), S(\varphi(g_2)) \ldots\ldots S(\varphi(g_n))$ is the totality of cosets of $S$ in $G'$.

Let $g' \in G'$

Then $\varphi(g) = g'$ for some $g \in G$, as $\varphi$ is onto

Let $g = hg_i$, Then $g' = \varphi(g) = \varphi(h)\,\varphi(g_i) \in S(\varphi(g_i))$

$\Rightarrow$ Every element of $G'$ is contained some $S(\varphi(g_i))$.

Hence the result

Alternatively we can show that $S(\varphi(g_1)) \ldots$
$S(\varphi(g_n))$ are all distinct

Suppose $S(\varphi(g_i)) = S(\varphi(g_j))$

Then $\varphi(g_i)(\varphi(g_j))^{-1} = \varphi(g_i g_j^{-1}) \in S$

Hence $g_i g_j^{-1} \in H$ as $H$ by correspondence theorem is preimage of $S$

$\Rightarrow Hg_i = Hg_j$

$\Rightarrow i = j$

Hence index of $S$ in $G'$ is $n$

**Problem:-** Let $G$ be a group and let $N$ be a normal subgroup of $G$. Suppose further that $L$ and $M$ are sub-groups of $G/N$. Then we can show that we can write $L$ in the form $H/N$, and $M$ in the form $K/N$, where $H$ and $K$ are sub-groups of $G$ containing $N$. Show also that if $L \subseteq M$, $H \subseteq K$; and if $L \trianglelefteq M$, $H \trianglelefteq K$. Show that if $L \subseteq M$ and $[M:L] = n < \infty$, then $[K:H] = n$.

**Solution:** By Corollary 2 of correspondence theorem

$$L = H/N \quad \text{and} \quad M = K/N.$$

Let $\varphi$ be natural homomorphism. We recall that

$$H = \{g \mid \varphi(g) \in L\} \text{ and } K = \{g \mid \varphi(g) \in M\}$$

Hence if $L \subseteq M$, $H \subseteq K$ follows immediately.

Now if $L \trianglelefteq M$

we consider the homomorphism

$$\psi : K \longrightarrow K/N \text{ defined by}$$

$$\psi(k) = \varphi(k)$$

i.e $\psi = \varphi|_K$

Clearly $\psi(K) = K/N$

and the preimage of $L$ is $H$, the preimage of $M$ is $K$. We can then conclude from the correspondence theorem that $H \trianglelefteq K$.

**Problem:-**

Let $N \trianglelefteq G$ and suppose $G/N$ is cyclic of order 6. Let $G/N = gp(Nx)$. Find all subgroups of $G/N$ and express them in the form Corollary 2 of correspondence theorem.

**Solution:-** Let $G/N = K$, Let $K_1 = \{N\}$, $K_2 = \{N, Nx^3\}$

$K_3 = \{N, Nx^2, Nx^3\}$ and $K_4 = K$.

These are all sub-groups of $K$.

To find the corresponding sub-groups Corollary 2

Let $\varphi : G \longrightarrow G/N$ be natural homomorphism i.e

$$\varphi(g) = Ng.$$

Let $G_i$ be the preimage of $K_i$ $i = 1,2,3,4$

$G_1 = \{g \mid \varphi(g) = N\} = \{g \mid Ng = N\} = \{g \mid g \in N\} = N$

$G_2 = \{g \mid \varphi(g) = N \text{ or } g = Nx^3\} = \{g \mid Ng = N \text{ or } Ng = Nx^3\} = N \cup Nx^3$

$G_3 = \{g \mid \varphi(g) = N \text{ or } \varphi(g) = Nx^2 \text{ or } \varphi(g) = Nx^4\} = N \cup Nx^2 \cup Nx^4$

$G_4 = \{g \mid \phi(g) \in K\} = G$. Then $G_i/N = K_i$ for $i = 1, 2, 3, 4$.

## The Subgroup isomorphism Theorem:

In homomorphism we were able to say (i) that the image of a homomorphism $\phi: G \to G'$ was essentially a factor group of $G$.

What can we say about the effect of $\phi$ on sub-groups? Let $H$ be a sub-group of $G$. Let $\psi = \phi|_H$ i.e. $\psi$ is the mapping of $H$ to $G'$ defined by

$$\psi(h) = \phi(h), \ h \in H$$

Then $\psi$ is a homomorphism of $H \to G'$ and so $\psi(H) = \phi(H) \cong H/(\ker\psi)$

Now if $\ker\phi = N = \{x \mid x \in G, \phi(x) = e'\}$.

Then $\ker\psi = \{x \mid x \in H \text{ and } \psi(x) = \phi(x) = e'\} = H \cap N$

So $\phi(H) = \psi(H) \cong H/(H \cap N)$. On the other hand, we know that $\phi(H)$ is a subset of $\phi(G)$ and

$$\phi(G) \cong G/N.$$

Our question is; what has $H/(H \cap N)$ got to do with $G/N$? It must be isomorphic to some subgroup of $G/N$. But which. This is what the sub-group isomorphism theorem is.

**Theorem** (Subgroup Isomorphism Theorem, also called the Second isomorphism theorem).

Let $A, B$ be sub-groups of a group $G$ with $A$ normal in $G$, Then.

(i) $\langle A, B\rangle = AB$ is a sub-group

(ii) $A \cap B$ is normal sub-group of $B$

(iii) $AB/A \cong B/A \cap B$

**Proof:-** (i) Let $A, B$ be sub-groups of $G$ and $A$ normal in $G$, then we first show that

$$\langle A, B\rangle = AB$$

Now each element of $\langle A, B\rangle$ is of the form

$x = a_1^{\alpha} b_1 a_2 b_2 a_3 b_3 \cdots a_k b_k^{\beta}$ where $\alpha, \beta$ are $0$ or $1$

$a_i \in A, b_i \in B$ $1 \le i \le k$

Since $A$ is normal in $G$

$\therefore \quad bab^{-1} \in A \qquad \forall a \in A, \forall b \in B.$

$\Rightarrow \quad bab^{-1} = a' \qquad$ for some $a' \in A.$

$\Rightarrow \quad ba = a'b$

So $\quad x = a_1^{\alpha} a_2' a_3' \cdots a_k' \, b_1 b_2 \cdots b_k^{\beta}$

$x = ab \qquad a \in A \qquad b \in B$

Thus $\quad x \in AB$

$\Rightarrow \langle A, B \rangle \subseteq AB$

Conversely each $ab \in AB$ is in $\langle A, B \rangle$.

So $\quad AB \subseteq \langle A, B \rangle$

Hence $\langle A, B \rangle = AB$

(ii) To show that $AB$ is a sub-group of $G$.

Let $x, y \in AB$

Then $x = ab \quad y = a'b' \qquad x, y \in AB$

$a, a' \in A, \; b, b' \in B$

$$xy^{-1} = ab.(a'b')^{-1}$$
$$= ab.(b'^{-1} a'^{-1})$$
$$= ab\, b'^{-1} a'^{-1}$$
$$= ab_1 a'^{-1} \qquad b_1 = b b'^{-1} \in B$$
$$= a a''' b_1 \qquad \because A \text{ is normal}$$
$$= a_1 b_1 \in AB \qquad \therefore b_1 a'^{-1} = a'' b_1$$
$$; \; a_1 = a a'' \in A$$

Hence $AB$ is sub-group of $G$

OR

Since $A$ is normal in $G$,

$\therefore \quad AB = BA$

$\Rightarrow AB$ is a sub-group of $G$ (by a previous theorem)

(ii)

Since $A$ & $B$ are sub groups of $G$, $A \cap B$ is a sub-group of $B$. To show that $A \cap B$ is normal in $B$

Let $x \in A \cap B$ & $b \in B$, then

$bxb^{-1} \in A \qquad \because A \trianglelefteq G$

$bxb^{-1} \in B \qquad \because B$ is subgp.

So that

$$bxb^{-1} \in A \cap B \qquad \forall x \in A \cap B, \forall b \in B$$

Hence $A \cap B$ is normal in $B$.

(iii) To prove $AB/A \cong B/A\cap B$

Every element of $AB/A$ is of the form

$abA = ba'A \quad \because A \trianglelefteq G$

$= bA \quad b \in B \quad \because ab = ba'$

Also let $D = A \cap B$

Then every element of $B/A\cap B = B/D$ is of the form $bD$, $b \in B$

We define a mapping $\psi : AB/A \longrightarrow B/D$ as

$\psi(bA) = bD$, $b \in B$

First we show that $\psi$ is well defined

For this suppose that

$bA = b'A$

We have to show that $bD = b'D$

Now.

$bA = b'A \Rightarrow b'^{-1}bA = A$

$\Rightarrow b'^{-1}b \in A$

But $b'^{-1}b \in B$

Hence

$b'^{-1}b \in A \cap B = D$ or $b'^{-1}b \in D$

So $b \in b'D$ But $b \in bD$

$\Rightarrow b'D \cap bD \neq \varphi$

Hence $bD = b'D$

So $\psi$ is obviously well defined

Next $\psi$ is obviously surjective

To see that $\psi$ is injective

Suppose that

$\psi(bA) = \psi(b'A)$ for some $b, b' \in B$

Then $bD = b'D$

$b'^{-1}bD = D$

i.e $b'^{-1}b \in D$

or $b'^{-1}b \in D = A \cap B$ So $b'^{-1}b \in A$

i.e $b \in b'A$

But $b \in bA$ also

Hence $bA = b'A \quad \because (bA) \cap (b'A) \neq \phi$

So $\psi$ is injective.

Lastly for $bA, b'A \in AB/A$, $bb' \in B$, we have

$$\psi(bA \cdot b'A) = \psi((bb')A)$$
$$= (bb')D = bD \cdot b'D$$
$$= \psi(bA) \cdot \psi(b'A)$$

So $\psi$ is homomorphism

As $\psi$ is bijective & homomorphism

So $AB/A \cong B/A \cap B$

**OR**

Let $\phi : B \longrightarrow AB/A$ be defined by

$$\phi(b) = bA$$

Then $\phi(bb') = bb'A = bA \cdot b'A$
$$= \phi(b) \cdot \phi(b')$$

$\Rightarrow \phi$ is homomorphism.

Let $hA \in AB/A$, where $h \in AB$

Now $h \in AB$ gives

$$h = a_1 b_1 \qquad a_1 \in A, b_1 \in B$$

Thus $hA = a_1 b_1 A = b_1 a' A$ $\quad \because A$ is normal
$$= b_1 A = \phi(b_1)$$

$\Rightarrow \phi$ is also onto i.e $\phi(B) = AB/A$

G By fundamental theorem of homomorphism

$\phi(B) \cong B/(\ker\phi)$ i.e $AB/A \cong B/(\ker\phi)$

(Shaum, outli)

$\ker\phi = \{x \mid x \in B;\ \phi(x) = e'\}$ $\qquad e' = eA = A$

$= \{x \mid x \in B,\ xA = A\}$ $\quad \because \phi(x) = xA$ & ~~$\phi(eA) = \phi(A) =$~~

If $xA = A$, then $x \cdot e = x \in A$ and if $x \in A$, $xA = A$.

Therefore $\ker\phi = \{x \mid x \in B, x \in A\} = B \cap A$.

Hence

$$\phi(B) \cong \frac{B}{B \cap A}$$

$$AB/A \cong B/B \cap A$$

G

**OR**

(Qazi, 83) Now $A$ is identity of $AB$.

So for any $b \in B$, $b \in \ker\varphi$

$\Leftrightarrow \varphi(b) = A$

$\Leftrightarrow bA = A$

$\Leftrightarrow b \in A$

$\Leftrightarrow b \in A \cap B$ ; since $b \in B$

i.e $b \in \ker\varphi \Leftrightarrow b \in A \cap B$

$\Rightarrow \ker\varphi = A \cap B$

Hence. $AB/A = B/A \cap B$

**Problem:-** Let $Q^*$ be the multiplicative grop of rationals. Let $N = \{1, -1\}$. Let $H$ be the sub-gp generated by $(1/2)$. Find $HN$, $HN/N$ and thereby the assertion of the sub-gp isomorphism theorem that

$HN/N = H/H \cap N$

**Solution:-** The elements of $H$ are the form $(\frac{1}{2})^r$, $r$ various integers.

$HN = \{x / x = hn,\ h \in H,\ n \in N\} = \{x / x = h \text{ or } x = -h, h \in H\}$
$= \{x / x = \pm(1/2)^r \text{ for all integers } r\}$

Cosets of $HN/N$ is of the form

$Nx = \{1, -1\}x = \{x, -x\}$ where $x \in HN$

Now if $x \in HN$, $x = \pm(1/2)^r$

Hence each coset is of the form $\{(1/2)^r, -(1/2)^r\}$.

Since $N(1/2).\ N(1/2)\ N(1/2) \cdots N(1/2) = N(1/2)^r$ and $(1/2)^r \in N(1/2)^r$, each coset of $HN/N$ is a power of $N(1/2)$.

Thus

$$gp(\{N(1/2)\}) = HN/N$$

Since $(\frac{1}{2})^r \notin N$ for $r \neq 0$

$\therefore$ $HN/N$ is the infinite cyclic gp.

Now $H \cap N = \{x / x = (\frac{1}{2})^r \text{ for some } r \text{ and } x = \pm 1\}$
$= \pm 1$

$H/(H\cap N) \cong H$

But $H$ is infinite cyclic.

Thus $H/(H\cap N) \cong HN/N$

**Problem** Let a group $G$ contains two normal sub-groups $M$ & $N$. Let $H$ be a sub-group of $G$. Prove that $HM/M \cong HN/N$ if $H\cap M = H\cap N$.

**Solution.** By sub-group isomorphism theorem

$$HM/M \cong H/H\cap M = H/H\cap N \cong HN/N$$

$$\Rightarrow \quad HN/N \cong HM/M$$

Remark: If $G/M$ has every element of order a power of 2, show that $H/H\cap N$ has every element of order a power of 2.

Solution

$$H/H\cap N = H/H\cap M \cong HM/M \subseteq G/M$$

Hence the result.

**Problem:** Let $H$ and $K$ be subgroups of $G$, $N \trianglelefteq G$ and $HN = KN$. Prove that $H/(H\cap N) \cong K/(K\cap N)$.

**Solution:**

Since $H, N$ are subgroups of $G$ and $N \trianglelefteq G$

$\therefore$ By sub-group isomorphism theorem

$$H/(H\cap N) \cong HN/N \longrightarrow ①$$

Also since $K, N$ are subgroups of $G$ and $N \trianglelefteq G$

$\therefore$ By sub-group isomorphism theorem

$$K/(K\cap N) \cong KN/N \longrightarrow ②$$

Since $HN = KN$

Therefore by ① & ②

$$H/(H\cap N) \cong K/(K\cap N)$$

($\because$ Two sub-groups isomorphic to same gp are isomorphic)

**Problem:-** Let $G \supseteq G_1 \supseteq G_2 \supseteq \{1\}$. Let $G_1 \trianglelefteq G$, $G_2 \trianglelefteq G_1$ and suppose $G/G_1$, $G_1/G_2$ and $G_2$ are abelian. Prove that if $H$ is any sub-group, then it has subgroup $H_1$ & $H_2$ such that $H_1 \trianglelefteq H$, $H_2 \trianglelefteq H_1$ and $H/H_1$, $H_1/H_2$ and $H_2$ are abelian.

**Solution:-** Let $H_1 = H \cap G_1$

Then by sub-group isomorphism theorem

$H_1 \trianglelefteq H$ and $H/H_1 \cong HG_1/G_1$

But $HG_1/G_1 \subseteq G/G_1$ and $G/G_1$ is abelian.

Hence $H/H_1$ is abelian.

Now consider $H_1$ as a subgroup of $G_1$

As $G_2 \trianglelefteq G_1$

$\therefore$ By sub-group isomorphism theorem

$H_1 \cap G_2 \trianglelefteq H_1$ and $H_1/(H_1 \cap G_2) \cong H_1G_2/G_2 \subseteq G_1/G_2$

Since $G_1/G_2$ is abelian.

So $H_1/(H_1 \cap G_2)$ is abelian.

Consequently we put $H_1 \cap G_2 = H_2$

Finally as $H_2 \subseteq G_2$ and $G_2$ is abelian.

Therefore $H_2$ is abelian.

**Problem:** Let $G \supseteq G_1 \supseteq \{1\}$ and $G_1 \trianglelefteq G$. Suppose $G/G_1$ and $G_1$ are abelian and $H$ is any sub-group of $G$ prove that there exists a sub-group $H_1$ of $H$ such that $H_1 \trianglelefteq H$ and $H/H_1$ and $H_1$ are abelian.

**Solution:**

Let $H_1 = H \cap G_1$

Then by the sub-group isomorphism theorem.

$H_1 \trianglelefteq H$ and $H/H_1 \cong HG_1/G_1$

But $\frac{HG_1}{G_1} \subseteq G/G_1$ and $G/G_1$ is abelian.

Therefore $H/H_1$ is abelian.

As $H_1 \subseteq G_1$ and $G_1$ is abelian

Therefore $H_1$ is abelian.

## Theorem (third isomorphism theorem).
(Factor of Factor theorem)

Let $H$ & $K$ be normal sub-groups of $G$ and $H \subseteq K$, then $K/H$ is normal sub-group of $G/H$ and

$$(G/H)/(K/H) \cong G/K$$

**Proof:**

Since $K$ is normal sub-group of $G$

$\therefore \quad g^{-1}kg \in K \quad \forall g \in G, \forall k \in K$

$\Rightarrow (Hg^{-1})(Hk)(Hg) = H(g^{-1}kg) \in HK$

$\Rightarrow (Hg^{-1})(Hk)(Hg) \in K/H \quad \forall Hg \in G/H, Hk \in K/H$

$\Rightarrow K/H$ is a normal sub-group of $G/H$

~~Thus G/K, G/H, K/H are meanigfu~~

Also the sub-group $H$, being a normal sub-gp. of $G$, is normal in any sub-group of $G$ containing $H$.
In particular $H$ is normal in $K$.
Thus $G/H$, $G/K$ and $K/H$ are all meaningful.

Define a mapping

$\phi : G/H \longrightarrow G/K$ by

$\phi(gH) = gK$

Then $\phi$ is surjective

Also

$$\begin{aligned}\phi(gH \cdot g'H) &= \phi(gg'H) \\ &= gg'K = gK \cdot g'K \\ &= \phi(gH) \cdot \phi(g'K)\end{aligned}$$

$\Rightarrow \phi$ is homomorphism.

Thus by first homomorphism theorem

$(G/H)/K' \cong G/K$

where $K' = \ker \phi$.

We show that $K' = K/H$

Obviously $K' \supseteq K/H$

Let $gH \in K'$

Let $kH \in K/H$
$\because K \subseteq G, H \subseteq G$
$\therefore \phi(kH) = kK = K$
$\Rightarrow kH \in K'$
$\therefore K' \supseteq K/H$

$\Rightarrow \quad \varphi(gH) = gK$

$= K \quad \because gH \in K'$

$\Rightarrow g \in K$

Hence $gH \in K/H$.

Thus $K' = K/H$.

Consequently

$(G/H)/(K/H) \cong G/K$

(Shanti 109) **Theorem:** Let $H$ be normal sub-group of $G$. $K$ is normal sub-group of $G$ containing $H$ iff $K/H$ is a normal sub-group of $G/H$.

**Solution:-** Let $K$ be sub-group of $G$ containing $H$ i.e.

$$H \subseteq K \subseteq G$$

Since $H$ is normal in $G$.

$\therefore \quad gH = Hg \quad \forall g \in G$.

and in particular

$gH = Hg \quad \forall g \in K$.

$\Rightarrow$ $H$ is a normal sub-group of $K$ and the symbol $K/H$ is meaningful.

Let $Hk_1, Hk_2 \in K/H$.

Then $Hk_1, Hk_2 \in G/H$.

we have

$k_1, k_2 \in K \Rightarrow k_1 k_2^{-1} \in K$ ($\because K$ is sub-group)

$\Rightarrow H(k_1 k_2^{-1}) \in K/H$

$\Rightarrow (Hk_1)(Hk_2^{-1}) \in K/H$.

$\Rightarrow (Hk_1)(Hk_2)^{-1} \in K/H$

$\Rightarrow K/H$ is a sub-group of $G/H$

~~Conversely~~ Consider any sub-gp of $G/H$ so that the members of this sub-group are some or all of the cosets of $H$. Let $K$ denote the set theoretic union of these cosets. We show that $K$ is a sub-group of $G$ containing $H$.

Let $x_1, x_2 \in K$ so that $Hx_1, Hx_2$ are members of the sub-group of $G/H$ in question

It follows that

$$(Hx_1)(Hx_2)^{-1} = (H x_1 x_2^{-1})$$

is as well a coset member of the sub-group $G/H$.

$\Rightarrow x_1 x_2^{-1} \in K$

Thus $K$ is sub-group.

Surely it contains $H$

Thus $K/H$ is sub-group of $G/H$

Now $K$ is a normal sub-group of $G$ iff

$$x^{-1}kx \in K \quad \forall x \in G, \forall k \in G$$

$$\Rightarrow (Hx^{-1})(Hk)(Hx) = H(x^{-1}kx) \in Hk$$

$$\forall Hx \in G/H, Hk \in K/H$$

$\Leftrightarrow K/H$ is a normal sub-group of $G/H$.

**Remark** A group $G$ is abelian if and only if $G$ coincides with its centre $\xi(G)$.

**Theorem:-** A group $G$ is abelian if and only if the factor group $G/\xi(G)$ is cyclic.

**Proof:-**

Let $G$ is abelian

Then $G = \xi(G)$.

So $G/\xi(G)$ is the trivial group and hence cyclic (The trivial or the identity group assumed to be generated by the empty set)

Conversely suppose that $G/\xi(G)$ is a cyclic group and $a\xi(G), a \in G$ is its generator.

Let $x, y \in G$

Then $x\xi(G), y\xi(G)$ belong to $G/\xi(G)$

So there exist integers $m, n$ such that

$x\xi(G) = a^m \xi(G)$, $y\xi(G) = a^n \xi(G)$

Thus $x = a^m z$, $y = a^n z'$ for some $z, z' \in \xi(G)$

$$xy = a^m z\, a^n z' = a^m a^n z z'$$

$$= a^n a^m z' z \quad (\because \xi(G) \text{ is abelian})$$

$= a^n z \cdot a^m z = yx.$

Consequently $G$ is abelian

**Theorem:** A group of order $p^2$, where $p$ is prime number, is abelian. (It is proved in p-groups).

**Problem:** If $H$ is a normal sub-group of $G$, then the mapping $f: G \longrightarrow G/H$ such that $f(a) = aH \; \forall a \in G$ is a homomorphism. Also determine its kernel.

**Solution:** Let $a, b \in G$

Now $f(a.b) = (ab)H$

$= (aH) \cdot (bH)$

$= f(a) \cdot f(b)$

$\Rightarrow f$ is a homomorphism.

We claim that $\ker f = K = H$

Let $k \in K = \ker f$

Then $f(k) = eH = H$ where $H$ is identity of $G/H$

Let $x \in K \subseteq G$

Then $f(x) = xH$

But $f(x) = H$

$\Rightarrow xH = H$

$\Rightarrow x \in H$

Since $x$ is an arbitrary element of $K$

$\therefore K \subseteq H \longrightarrow (1)$

Again let $y \in H$

Then $yH = H$

$\Rightarrow f(y) = H$

$\Rightarrow y \in \ker f = K$

$\Rightarrow H \subseteq K \longrightarrow (2)$

By (1) & (2)

$K = H.$

## ~~Endomorphisms~~ & ~~Automorphisms~~

(2021'84) **Problem:-** For $a, b \in R$, $a \neq 0$, define $f: R \to R$ by $f_{ab}(x) = ax+b$. Let $G = \{f_{ab} | a, b \in R, a \neq 0\}$ and $N = \{f_{1b} \in G\}$. Prove that $N$ is a normal sub-group of $G$ and $G/N \cong$ the group of non-zero real numbers under multiplication.

**Solution:-** $f_{10}(x) = x$ shows that the identity mapping $I_R \in G$. So $G$ is non empty.

Let $a, b, c, d \in R$ with $a \neq 0$, and $c \neq 0$.

Then $f_{ab}[f_{cd}(x)] = f_{ab}(cx+d) = a(cx+d)+b$

$= acx + ad + b$

Thus $f_{ab} f_{cd} = f_{ac,\, ad+b}$ $\longrightarrow$ ①

Under this operation $G$ is group with $(f_{ab})^{-1} = f_{a^{-1}, -a^{-1}b}$ and $f_{10}$ as identity.

Clearly $f_{10} \in N$, So $N$ is non-empty.

Let $f_{ac,\, ad+b} = f_{10}$
$\Rightarrow ac = 1 \Rightarrow c = a^{-1}$
$ad + b = 0$
$\Rightarrow d = -a^{-1}b$
Hence
$(f_{ab})^{-1} = f_{a^{-1}, -a^{-1}b}$

Let $f_{1b}, f_{1d} \in N$

Then $f_{1b}(f_{1d})^{-1} = f_{1b} f_{1,-d}$

$= f_{1, -d+b} \in N$

So $N$ is a sub-group of $G$.

Again if $f_{ab} \in G$, $f_{1c} \in N$

$$f_{ab} f_{1c} (f_{ab})^{-1} = f_{a,\, ac+b}\, f_{a^{-1}, -a^{-1}b}$$

$$= f_{1, ac} \in N$$

Hence $N$ is normal sub-group of $G$.

Finally let $K$ be the group of non-zero real numbers under multiplication.

Define $\theta: G \to K$ by.

$\theta(f_{ab}) = a \quad \forall\, f_{ab} \in G$

Clearly $\theta$ is onto

$\theta(f_{ab} f_{cd}) = \theta(f_{ac,\, ad+b}) = ac = \theta(f_{ab})\theta(f_{cd})$

$\Rightarrow \theta$ is a homomorphism.

$f_{ab} \in \ker\theta \Leftrightarrow \theta(f_{ab}) = $ identity of $K$

$\Leftrightarrow a = 1$

$\Leftrightarrow$ $fab = f1b$

$\Leftrightarrow$ $fab \in N$

Hence $\ker \theta = N$

By fundamental theorem of homomorphism

$$G/N \cong K$$

# Commutator or Derived Sub-groups

**Commutator** Let $G$ be a group and $a, b \in G$, then the element $aba^{-1}b^{-1}$ is called the commutator of elements $a$ & $b$ and is denoted by $[a\,b]$

**Derived or First Derived Group**

The group generated by all the commutators $[a, b]$, $a, b \in G$ is called the commutator sub-gp of $G$ or the first derived group of $G$ and is denoted by $G'$ or $G^c$

**Theorem** prove that the inverse of a commutator is a commutator

**Proof:** $[a\,b] = aba^{-1}b^{-1} = z$ say.

So $z^{-1} = (aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1} = [b\,a]$

**Theorem** The following commutator identities hold in a group:

i) $[ab]^{-1} = [b\,a]$

ii) $[ab, c] = [b\,.\,c]^a [a, c]$

iii) $[a\ bc] = [a\ b]\,[a, c]^b$

iv) $[a, b^{-1}] = [b\ a]^{b^{-1}}$ and

$[a^{-1}\ b] = [b\ a]^{a^{-1}}$ $\forall\, a, b, c \in G$

Here $x^a$ denotes the conjugate $axa^{-1}$ of $x$.

**Proof:-**

(i) Since $[a\ b] = aba^{-1}b^{-1}$

$\Rightarrow [a\ b]^{-1} = (aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1} = [b\ a]$

(ii) $[ab, c] = abc\,(ab)^{-1}c^{-1}$

$= abcb^{-1}a^{-1}c^{-1}$

$= a(bcb^{-1})(ca)^{-1}$

$= a(bcb^{-1})c^{-1}c\,(ca)^{-1}$

$= a(bcb^{-1}c^{-1})a^{-1}ac\,(ca)^{-1}$

$= a(bcb^{-1}c^{-1})a^{-1}\,aca^{-1}c^{-1}$

$= [b\ c]^a\,[a\ c]$

**Remark:** ①: The commutator of two elements $a, b$ is identity iff $a$ & $b$ commute.
②: Every element conjugate to a commutator is a commutator.

iii) $[a\ bc] = a(bc)\bar{a}^1(bc)^{-1}$
$= a\,bc\,\bar{a}^1\bar{c}^1 b^{-1}$
$= ab\bar{a}^1 a\,c\,\bar{a}^1\bar{c}^1 b^{-1}$
$= ab\bar{a}^1 b^{-1} b(ac\bar{a}^1\bar{c}^1)b^{-1}$
$= [a\ b][a\ c]^b$

(iv) $[a\ b^{-1}] = ab^{-1}\bar{a}^1 b$
$= b^{-1}b\,ab^{-1}\bar{a}\,b$
$= [b\ a]^{b^{-1}}$

and similarly $[\bar{a}^1\ b] = [b\ a]^{\bar{a}^1}$

**Theorem:** A group $G$ is abelian iff $G' = \{e\}$ i.e iff $[a\ b] = e \ \forall a, b \in G$

**Proof:** Suppose $G$ is abelian and $a, b \in G$. Then $ab = ba$
$\therefore [a\ b] = ab\bar{a}^1 b^{-1} = ba\bar{a}^1 b^{-1} = e$.
$\Rightarrow G'$ is sub-gp of $G$ generated by $e$ and $G' = e$
If $G' = e$, then in particular any commutator $[a\ b] = ab\bar{a}^1 b^{-1} = e$
$\Rightarrow ab\bar{a}b^{-1} = e$
$\Rightarrow ab = ba$
$\Rightarrow G$ is abelian.

**Theorem** Let $G$ be a group. Then
(a) the derived group $G'$ is a normal sub-gp of $G$
(b) the factor group $G/G'$ is abelian
(c) If $K$ is normal sub gp of $G$ s.that $G/K$ is abelian then $K \supseteq G'$

**Proof** To show that $G'$ is normal in $G$, we have to show that for each commutator $q$ in $G'$ and $g \in G$, $gq\bar{g}^1 \in G'$

$q = ab\bar{a}^1 b^{-1}$ $\quad a, b \in G$

So $gq\bar{g}^1 = g\,ab\bar{a}^1 b^{-1}\bar{g}^1$
$= ga\bar{g}^1 g\,b\bar{g}g\,\bar{a}^1\bar{g}^1 g\,b^{-1}\bar{g}^1$
$= [a^g,\ b^g]$ is an element of $G'$

Hence $G'$ is normal in $G$.

(b) Let $aG', bG' \in G/G'$.

Then

$$[aG'\ bG'] = aG'bG'(aG')^{-1}(bG')^{-1}$$
$$= aba^{-1}b^{-1}G'$$
$$= G' \quad \because aba^{-1}b^{-1} \in G'$$

Thus $[aG'\ bG'] = G'$, identity of $G/G'$

$\Rightarrow$ $G/G'$ is abelian

OR

$$aba^{-1}b^{-1}G' = G'$$
$$\Rightarrow (ab)G' = baG'$$
$$\Rightarrow aG'.bG' = bG'.aG' \quad \text{i.e } G/G' \text{ is abelian.}$$

(c) Suppose that for any normal sub-group $K$ of $G$ $G/K$ is abelian.

To show that $G' \subseteq K$. We have only to prove that every commutator $[a, b] \in K$.

Since $G/K$ is abelian

$$\therefore ak.bk = bk.ak$$
$$(ak.bk)\, a^{-1}k\, b^{-1}k = k$$
$$(aba^{-1}b^{-1})\, k = k$$
$$\Rightarrow [a\ b] \in K$$

Hence $G' \subseteq K$

Shaum (11.6)

**Theorem:** If $G$ is abelian, then $G/G' \cong G$

**Proof:-**

$G'$ is the sub-group generated by the Commutators $aba^{-1}b^{-1}$, $a, b \in G$

As $G$ is abelian

Therefore $aba^{-1}b^{-1} = abb^{-1}a^{-1} = e$

$\Rightarrow$ $G'$ is sub-gp generated by $e$

Since any product $e$ and its inverse is again $e$, $G' = \{e\}$

Let $\phi$ be the natural homomorphism of $G$ to $G/G'$

To show $\phi$ is isomorphism we need only show that it is one-one.

Let $\phi(a_1) = \phi(a_2)$

$\Rightarrow ea_1 = ea_2$

$\Rightarrow a_1 = a_2$

$\Rightarrow \phi$ is $(1-1)$

Thus $\phi$ is an isomorphism.

Shanti (88) **Theorem** A quotient group $G/H$ is abelian iff $H$ contains the commutator sub-group of $G$.

**Proof**: Suppose that $G/H$ is abelian and $aH, bH \in G/H$

Then
$$aH \cdot bH = bH \cdot aH$$
$$(ab)H = (ba)H$$
$$H = (ab)^{-1}(ba)H$$
$$= (b^{-1}a^{-1}ba)H.$$

$\Rightarrow b^{-1}a^{-1}ba \in H \quad \forall a, b \in G$

$\Rightarrow H$ contains every commutator and so contains $G'$

Conversely let $G'$ is contained in $H$. i.e

$aba^{-1}b^{-1} \in H \quad \forall a, b, G$

$\Rightarrow aba^{-1}b^{-1}H = H$

or $Haba^{-1}b^{-1} = H$

$Hab = Hba$

$Ha \cdot Hb = Hb \cdot Ha$

$\Rightarrow G/H$ is abelian.

Shaum (116) **Theorem** If $H$ is a subgroup of $G$ containing $G'$, then $H \trianglelefteq G$.

**Proof** Let $h \in H$ and consider $g^{-1}hg$

Now $g^{-1}hgh^{-1}$ is a commutator and belongs to $G'$ and thus to $H$

Therefore $g^{-1}hgh^{-1}h = g^{-1}hg \in H$ and $H \trianglelefteq G$.

OR

Let $\phi: G \to G/G'$ be natural homomorphism, then $\phi$ is onto. $G/G'$ is abelian. Any sub-group of $G/G'$

is therefore normal and thus $\phi(H) = S$ is normal in $G/G'$. By the correspondence, $H$ is the preimage of $S$. Hence using the correspondence theorem once more, $H$ is normal in $G$.

## P- Group

Let P be a prime. A gp G is a a P-group if every element in G has order a a power of the prime p.

OR

Let p be a prime. A finite p-group is is a group of order $p^\alpha$, $\alpha \geq 1$

**P-Sub-group** A sub-gp of a gp G is a P-sub-group of G if the sub-group is itself a P-group.

OR

Let G be a finite group of order n and p a prime divisor of of n. A sub-gp H of G is called a p-sub-gp if H is p-sub-gp.

## Sylow p-Sub-group of G

Let G be a group of order n and p a prime divisor of n. A sub-group H of G is said to be a Sylow p-sub-gp of G if H has order $p^\alpha$ where $p^\alpha$ divides n but $p^{\alpha+1}$ does not divide n.

OR

A sub-gp H of a finite group G is a Sylow

p-group iff the order of H is a power of p and the index of H is prime to p.

## Double Coset

Let H & K be sub-groups of a group G and (a) an arbitrary element of G. Then set $HaK = \{hak, h \in H, k \in K\}$ is called a double coset in G modulo (H, K) determined by (a)

**Theorem** Let H, K be finite sub-groups of a group G. The double coset HaK contain $mn/q$ elements, where m, n and q are the orders of the sub-groups H, K and $Q = H \cap aKa^{-1}$

**Theorem** (Sylow, 2nd Theorem)
Any two Sylow p-sub-groups of a group are conjugate.

**Proof** # Let G be a group of order n & H, K be any two Sylow p-sub-groups each of order $p^{\alpha}$ in G. Then

$$n = p^{\alpha} m \qquad \& \ (P, m) = 1$$

$\because$ G double cosets form a partition of G

③

$G = \bigcup_{i=1}^{r} Ha_iK \qquad a_i \in G$

$\Rightarrow |G| = |Ha_1K| + |Ha_2K| + \cdots + |Ha_rK|$.

$\because |Ha_iK| = \dfrac{p^\alpha \cdot p^\alpha}{q_i}$

where $q_i$ is the order of $H \cap a_iKa_i^{-1}$

$\Rightarrow |G| = \dfrac{p^\alpha \cdot p^\alpha}{q_1} + \dfrac{p^\alpha p^\alpha}{q_2} + \cdots + \dfrac{p^\alpha p^\alpha}{q_r}$

$$n = \sum_{i=1}^{r} \frac{p^\alpha p^\alpha}{q_i} \longrightarrow ①$$

Dividing by $p^\alpha$

$$m = \sum_{i=1}^{r} p^\alpha / q_i \longrightarrow (2)$$

Now $q_i = |H \cap a_iKa_i^{-1}|$

$\Rightarrow q_i$ is order the intersection of two $P$-subgps

$\Rightarrow$ each term in $q_i$ is some power of $p$

$\Rightarrow$ R.H.S of ② is either multiple of $p$ or $1$.

$\because$ left hand side is not divisible by $p$

$\therefore p^\alpha / q_i = 1$ for at least one $i = 1, 2, \ldots r$.

Let $\dfrac{p^\alpha}{q_1} = 1 \Rightarrow p^\alpha = q_1$

$\Rightarrow p^\alpha = q_1 = |H \cap a_1 K a_1^{-1}|$

But $H \cap a_1 K a^{-1}$ is a sub-gp of $H$ of the same order as that of $H$ so

$$H = H \cap a_1 K a^{-1}$$

$$\Rightarrow H \subseteq a_1 K a_1^{-1}$$

As $|H| = |K| = |a_1 K a_1^{-1}|$

$$\Rightarrow H = a_1 K a_1^{-1}$$

Hence $H$ and $K$ are conjugate.

<u>Corollary</u> A finite gp $G$ has a unique Sylow $p$-sub-group $H$ iff $H$ is normal in $G$.

<u>Proof</u> # Let $H$ be unique Sylow $p$-sub-gp and $a \in G$. Then $aHa^{-1}$ is also a Sylow $p$-sub-gp by above Theorem

But $H$ is unique Sylow $p$-sub-gp of $G$

$$\Rightarrow aHa^{-1} = H$$

$$\Rightarrow aH = Ha \quad \forall a \in G$$

$\Rightarrow$ $H$ is normal in $G$.

Conversely if $H$ is normal in $G$, Then

$$aHa^{-1} = H \quad \forall a \in G$$

$\because$ all Sylow $p$-sub-gps are of the form $aHa^{-1}$, $a \in G$ and all these coincide

with H. Thus H is unique Sylow - p - sub group

**Theorem #** The number $k$ of Sylow p- sub-groups of a finite group is Congruent to 1 mod p and is a factor of the order of the gp.

**Proof #** Let $H$ be a Sylow p- Sub-group of $G$. Let $n$ be the order of $G$. Since any two sub-gps are conjugate, the no Sylo - p- sub-gp of $G$ is equal to the no of sub-groups in a conjugacy class of $H$ and no of Conjugacy class is equal to the index of the normaliser $(N_G(H) = N)$ of $H$ in $G$.

Let $|H| = p^\alpha$ $\quad |N| = n_1$

$(G : N) = k$ we show that

$$K \equiv 1 \bmod p$$

double coset decomposition modulo $(N, H)$ of $G$.

$$G = \bigcup_{i=1}^{s} Na_iH, \quad a_i \in G$$

Then $$n = \sum_{i=1}^{s} \frac{n_1 \cdot p^\alpha}{q_i} \longrightarrow (1)$$

where $q_i = |N \cap a_iHa_i^{-1}|$

$\Rightarrow$ $q_i$ is a power of $p$ because it is the order of a sub-group of a p- group $a_iHa_i^{-1}$. Hence

Dividing (1) by $n_1$.

⑥

$$\frac{n}{n_1} = \sum_{i=1}^{r} p^{\alpha}/q_i$$

$$k = \sum_{i=1}^{r} p^{\alpha}/q_i \longrightarrow ②$$

When $(G:N) = k$.

each term on R.H.S of ② is multiple of p or unity.

However one term among the double cosets $Na_iH$, ~~$NeH$~~ is such that $NeH = NH$ let $a_1 = e$

$$Na_1H = NH = H \quad \text{as } H \subseteq N$$

and $N \cap H = H$

so $q_1 = p^{\alpha} = |H|$ $[\because q_1 = |N \cap a_1 H a_1^{-1}| = |N \cap H| = |H| = p^{\alpha}]$

and corresponding term in ② is

$$p^{\alpha}/q_1 = 1$$

and

$$k = 1 + \sum_{i=2}^{r} p^{\alpha}/q_i \longrightarrow ③$$

and no other term in ③ is unity

because if for some $j > 1$, $p^{\alpha}/q_j = 1$ in ③

, then $q_j = p^{\alpha}$

and.

$N \cap a_j H a_j^{-1}$ being sub-gp of $a_j H a_j^{-1}$

and $|N \cap a_j H a_j^{-1}| = q_j = p^{\alpha}$

$\Rightarrow a_j H a_j^{-1} = N \cap a_j H a_j^{-1}$

So that

$$a_j H a_j^{-1} \subseteq N$$

∵ a Sylow $p$-sub-gp $H$ of $G$ is a Sylow $p$-sub-gp of any Sub gp Containing $H$

∴ $H$ is a Sylow $p$-sub-gp of $N$.

But $H$ is normal in its normaliser $N$.

So $H$ is unique Sylow $p$-sub-gp of $N$.

So

$$H = a_j H a_j^{-1}$$

Thus $a_j \in N$

$$\Rightarrow \quad N a_j H = N H = N a_1 H \qquad \because a_j \in N$$

$\Rightarrow j = 1$, a Contradiction.

Hence no other term in R.H.S of (3) except 1st is unity.

So $\sum_{i=2}^{k} p^{\alpha}/q_i$ is a multiple of $p$

because each term is multiple of $p$

The

$$k = 1 + l p \qquad \text{for Some integer } l.$$

$$\Rightarrow K \equiv 1 \bmod p$$

∴ $k$ is the index of a Sub-group of $G$, $k$ is a factor of the order of $G$. Proved!